

# ESSENTIAL VARIABLES AND POSITIONS IN TERMS

SLAVCHO SHTRAKOV

**ABSTRACT.** The paper deals with  $\Sigma$ -composition of terms, which allows us to extend the derivation rules in formal deduction of identities. The concept of essential variables and essential positions of terms with respect to a set of identities is a key step in the simplification of the process of formal deduction.  $\Sigma$ -composition of terms is defined as replacement between  $\Sigma$ -equal terms. This composition induces  $\Sigma R$ -deductively closed sets of identities. In analogy to balanced identities we introduce and investigate  $\Sigma$ -balanced identities for a given set of identities  $\Sigma$ .

## 1. INTRODUCTION

Let  $\mathcal{F}$  be any finite set, the elements of which are called *operation symbols*. Let  $\tau : \mathcal{F} \rightarrow \mathbb{N}$  be a mapping into the non-negative integers; for  $f \in \mathcal{F}$ , the number  $\tau(f)$  will denote the *arity* of the operation symbol  $f$ . The pair  $(\mathcal{F}, \tau)$  is called a *type* or *signature*. If it is obvious what the set  $\mathcal{F}$  is, we will write “*type*  $\tau$ ”. The set of symbols of arity  $p$  is denoted by  $\mathcal{F}_p$ .

Let  $X$  be a finite set of variables, and let  $\tau$  be a type with the set of operation symbols  $\mathcal{F} = \cup_{j \geq 0} \mathcal{F}_j$ . The set  $W_\tau(X)$  of *terms of type*  $\tau$  with variables from  $X$  is the smallest set such that

- (i)  $X \cup \mathcal{F}_0 \subseteq W_\tau(X)$ ;
- (ii) if  $f$  is an  $n$ -ary operation symbol and  $t_1, \dots, t_n$  are terms, then the “string”  $f(t_1 \dots t_n)$  is a term.

An algebra  $\mathcal{A} = \langle A; \mathcal{F}^{\mathcal{A}} \rangle$  of type  $\tau$  is a pair consisting of a set  $A$  and an indexed set  $\mathcal{F}^{\mathcal{A}}$  of operations, defined on  $A$ . If  $f \in \mathcal{F}$ , then  $f^{\mathcal{A}}$  denotes a  $\tau(f)$ -ary operation on the set  $A$ . We denote by  $\text{Alg}(\tau)$  the class of all algebras of type  $\tau$ . If  $s, t \in W_\tau(X)$ , then the pair  $s \approx t$  is called an identity of type  $\tau$ .  $\text{Id}(\tau)$  denotes the set of all identities of type  $\tau$ .

An identity  $t \approx s \in \text{Id}(\tau)$  is satisfied in the algebra  $\mathcal{A}$ , if the term operations  $t^{\mathcal{A}}$  and  $s^{\mathcal{A}}$ , induced by the terms  $t$  and  $s$  on the algebra  $\mathcal{A}$  are equal, i.e.,  $t^{\mathcal{A}} = s^{\mathcal{A}}$ . In this case we write  $\mathcal{A} \models t \approx s$  and if  $\Sigma$  is a set of identities of type  $\tau$ , then  $\mathcal{A} \models \Sigma$  means that  $\mathcal{A} \models t \approx s$  for all  $t \approx s \in \Sigma$ .

---

2000 *Mathematics Subject Classification.* Primary: 08B05; Secondary: 08A02, 03C05, 08B15.

*Key words and phrases.* Composition of terms, Essential position in a term, Globally invariant congruence, Stable variety.

Let  $\Sigma$  be a set of identities. For  $t, s \in W_\tau(X)$  we write  $\Sigma \models t \approx s$  if, given any algebra  $\mathcal{A}$ ,  $\mathcal{A} \models \Sigma \Rightarrow \mathcal{A} \models t \approx s$ .

The operators  $Id$  and  $Mod$  are defined for classes of algebras  $K \subseteq Alg(\tau)$  and for sets of identities  $\Sigma \subseteq Id(\tau)$  as follows

$$Id(K) := \{t \approx s \mid \mathcal{A} \in K \Rightarrow \mathcal{A} \models t \approx s\}, \text{ and} \\ Mod(\Sigma) := \{\mathcal{A} \mid t \approx s \in \Sigma \Rightarrow \mathcal{A} \models t \approx s\}.$$

The set of fixed points with respect to the closure operators  $IdMod$  and  $ModId$  form complete lattices  $\mathcal{L}(\tau)$  and  $\mathcal{E}(\tau)$  of all varieties of type  $\tau$  and of all equational theories (logics) of type  $\tau$ .

In [1] deductive closures of sets of identities are used to describe some elements of these lattices. We will apply the concept of  $\Sigma$ -compositions of terms to study the lattices  $\mathcal{L}(\tau)$  and  $\mathcal{E}(\tau)$ . We use the concept of essential variables, as defined in [5] and therefore we consider such variables with respect to a given set of identities, which is a fully invariant congruence.

In Section 2 we investigate the concept of  $\Sigma$ -essential variables and positions. The fictive (non-essential) variables and positions are used to simplify the deductions of identities in equational theories. We introduce  $\Sigma$ -composition of terms for a given set  $\Sigma$  of identities.

In Section 3 we describe the closure operator  $\Sigma R$  in the set of all identities of a given type, which generate extensions of fully invariant congruences. The varieties which satisfy  $\Sigma R$ -closed sets are fully invariant congruences and they are called stable. The stable varieties are compared to solid ones [2, 4, 6].

In Section 4 we introduce and study  $\Sigma$ -balanced identities and prove that  $\Sigma$ -balanced property is closed under  $\Sigma R$ -deductions.

## 2. COMPOSITIONS OF TERMS

If  $t$  is a term, then the set  $var(t)$  consisting of those elements of  $X$  which occur in  $t$  is called the set of *input variables (or variables)* for  $t$ . If  $t = f(t_1, \dots, t_n)$  is a non-variable term, then  $f$  is the *root symbol (root)* of  $t$  and we will write  $f = root(t)$ . For a term  $t \in W_\tau(X)$  the set  $Sub(t)$  of its subterms is defined as follows: if  $t \in X \cup \mathcal{F}_0$ , then  $Sub(t) = \{t\}$  and if  $t = f(t_1, \dots, t_n)$ , then  $Sub(t) = \{t\} \cup Sub(t_1) \cup \dots \cup Sub(t_n)$ .

The *depth* of a term  $t$  is defined inductively: if  $t \in X \cup \mathcal{F}_0$  then  $Depth(t) = 0$ ; and if  $t = f(t_1, \dots, t_n)$ , then  $Depth(t) = \max\{Depth(t_1), \dots, Depth(t_n)\} + 1$ .

**Definition 2.1.** Let  $r, s, t \in W_\tau(X)$  be three terms of type  $\tau$ . By  $t(r \leftarrow s)$  we will denote the term, obtained by simultaneous replacement of every occurrence of  $r$  as a subterm of  $t$  by  $s$ . This term is called the *inductive composition* of the terms  $t$  and  $s$ , by  $r$ . In particular,

- (i)  $t(r \leftarrow s) = t$  if  $r \notin Sub(t)$ ;
- (ii)  $t(r \leftarrow s) = s$  if  $t = r$ , and

- (iii)  $t(r \leftarrow s) = f(t_1(r \leftarrow s), \dots, t_n(r \leftarrow s))$ , if  $t = f(t_1, \dots, t_n)$  and  $r \in \text{Sub}(t)$ ,  $r \neq t$ .

If  $r_i \notin \text{Sub}(r_j)$  when  $i \neq j$ , then  $t(r_1 \leftarrow s_1, \dots, r_m \leftarrow s_m)$  means the inductive composition of  $t, r_1, \dots, r_m$  by  $s_1, \dots, s_m$ . In the particular case when  $r_j = x_j$  for  $j = 1, \dots, m$  and  $\text{var}(t) = \{x_1, \dots, x_m\}$  we will briefly write  $t(s_1, \dots, s_m)$  instead of  $t(x_1 \leftarrow s_1, \dots, x_m \leftarrow s_m)$ .

Any term can be regarded as a tree with nodes labelled as the operation symbols and its leaves labelled as variables or nullary operation symbols. Often the tree of a term is presented by a diagram of the corresponding term as it is shown by Figure 1.

Let  $\tau$  be a type and  $\mathcal{F}$  be its set of operation symbols. Denote by  $\maxar = \max\{\tau(f) \mid f \in \mathcal{F}\}$  and  $N_{\mathcal{F}} := \{m \in \mathbb{N} \mid m \leq \maxar\}$ . Let  $N_{\mathcal{F}}^*$  be the set of all finite strings over  $N_{\mathcal{F}}$ . The set  $N_{\mathcal{F}}^*$  is naturally ordered by  $p \preceq q \iff p$  is a prefix of  $q$ . The Greek letter  $\varepsilon$ , as usual denotes the empty word (string) over  $N_{\mathcal{F}}$ .

To distinguish between different occurrences of the same operation symbol in a term  $t$  we assign to each operation symbol a position, i.e., an element of a given set. Usually positions are finite sequences (strings) of natural numbers. Each position is assigned to a node of the tree diagram of  $t$ , starting with the empty sequence  $\varepsilon$  for the root and using the integers  $j$ ,  $1 \leq j \leq n_i$  for the  $j$ -th branch of an  $n_i$ -ary operational symbol  $f_i$ . So, let the position  $p = a_1 a_2 \dots a_s \in N_{\mathcal{F}}^*$  be assigned to a node of  $t$  labelled by the  $n_i$ -ary operational symbol  $f_i$ . Then the position assigned to the  $j$ -th child of this node is  $a_1 a_2 \dots a_s j$ . The set of positions of a term  $t$  is denoted by  $\text{Pos}(t)$  and it is illustrated by Example 2.1.

Thus we have  $\text{Pos}(t) \subseteq N_{\mathcal{F}}^*$ .

Let  $t \in W_{\tau}(X)$  be a term of type  $\tau$  and let  $\text{sub}_t : \text{Pos}(t) \rightarrow \text{Sub}(t)$  be the function which maps each position in a term  $t$  to the subterm of  $t$ , whose root node occurs at that position.

**Definition 2.2.** Let  $t, r \in W_{\tau}(X)$  be two terms of type  $\tau$  and  $p \in \text{Pos}(t)$  be a position in  $t$ . The positional composition of  $t$  and  $r$  on  $p$  is the term  $s := t(p; r)$  obtained from  $t$  by replacing the term  $\text{sub}_t(p)$  by  $r$  on the position  $p$ , only.

**Example 2.1.** Let  $\tau = (2)$ ,  $t = f(f(x_1, f(f(f(x_1, x_2), x_2), x_3)), x_4)$  and  $u = f(x_4, x_1)$ . The positions of  $t$  and  $u$  are written on their nodes in Figure 1. Then the positional composition of  $t$  and  $u$  on the position  $121 \in \text{Pos}(t)$  is  $t(121; u) = f(f(x_1, f(f(x_4, x_1), x_3)), x_4)$  and  $\text{sub}_t(121) = f(f(x_1, x_2), x_2)$ .

**Remark 2.1.** The positional composition has the following properties:

1. If  $\langle \langle p_1, p_2 \rangle, \langle t_1, t_2 \rangle \rangle$  is a pair with  $p_1 \not\preceq p_2$  &  $p_2 \not\preceq p_1$ , then

$$t(p_1, p_2; t_1, t_2) = t(p_1; t_1)(p_2; t_2) = t(p_2; t_2)(p_1; t_1);$$

2. If  $S = \langle p_1, \dots, p_m \rangle$  and  $T = \langle t_1, \dots, t_m \rangle$  with

$$(\forall p_i, p_j \in S) (i \neq j \Rightarrow p_i \not\preceq p_j \text{ \& } p_j \not\preceq p_i)$$

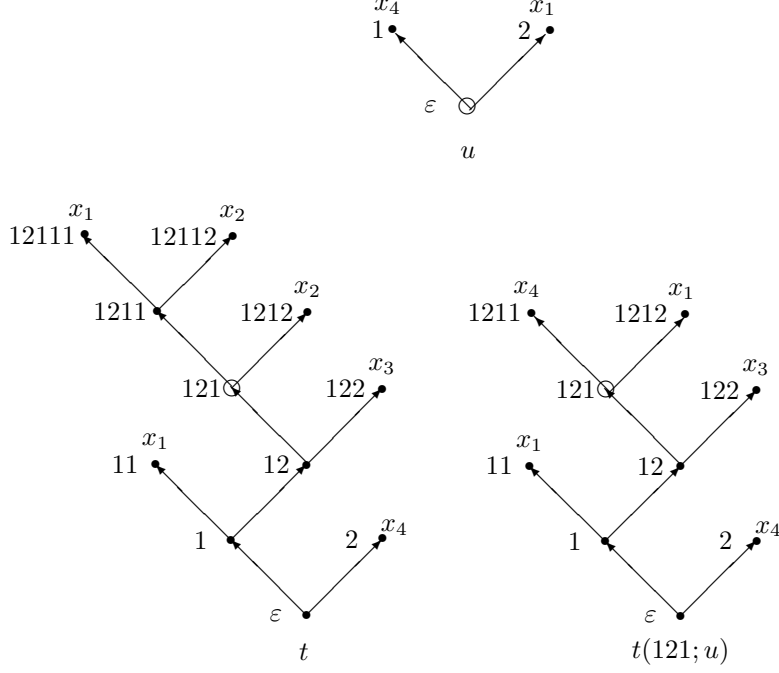


FIGURE 1. Positional composition of terms

and  $\pi$  is a permutation of the set  $\{1, \dots, m\}$ , then

$$t(p_1, \dots, p_m; t_1, \dots, t_m) = t(p_{\pi(1)}, \dots, p_{\pi(m)}; t_{\pi(1)}, \dots, t_{\pi(m)}).$$

3. If  $t, s, r \in W_\tau(X)$ ,  $p \in Pos(t)$  and  $q \in Pos(s)$ , then  $t(p; s(q; r)) = t(p; s)(pq; r)$ .
4. Let  $s, t \in W_\tau(X)$  and  $r \in Sub(t)$  be terms of type  $\tau$ . Let  $\{p_1, \dots, p_m\} = \{p \in Pos(t) \mid sub_t(p) = r\}$ . Then we have

$$t(p_1, \dots, p_m; s) := t(p_1; s)(p_2; s), \dots, (p_m; s) = t(r \leftarrow s),$$

which shows that any inductive composition can be represented as a positional one. On the other side there are examples of positional compositions which can not be realized as inductive compositions.

**Definition 2.3.** Let  $\Sigma \subseteq Id(\tau)$ ,  $t \in W_\tau(X_n)$  be an  $n$ -ary term of type  $\tau$ ,  $\mathcal{A} = \langle A, \mathcal{F} \rangle$  be an algebra of type  $\tau$  and let  $x_i \in var(t)$  be a variable which occurs in  $t$ .

(i) [5] The variable  $x_i$  is called *essential* for  $t$  with respect to the algebra  $\mathcal{A}$  if there are  $n+1$  elements  $a_1, \dots, a_{i-1}, a, b, a_{i+1}, \dots, a_n \in A$  such that

$$t^{\mathcal{A}}(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) \neq t^{\mathcal{A}}(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n).$$

The set of all essential variables for  $t$  with respect to  $\mathcal{A}$  will be denoted by  $Ess(t, \mathcal{A})$ .  $Fic(t, \mathcal{A})$  denotes the set of all variables in  $var(t)$ , which are not essential with respect to  $\mathcal{A}$ , called fictive ones.

(ii) A variable  $x_i$  is said to be  $\Sigma$ -essential for a term  $t$  if there is an algebra  $\mathcal{A}$ , such that  $\mathcal{A} \models \Sigma$  and  $x_i \in Ess(t, \mathcal{A})$ . The set of all  $\Sigma$ -essential variables for  $t$  will be denoted by  $Ess(t, \Sigma)$ . If a variable is not  $\Sigma$ -essential for  $t$ , then it is called  $\Sigma$ -fictive for  $t$ .  $Fic(t, \Sigma)$  denotes the set of all  $\Sigma$ -fictive variables for  $t$ .

**Proposition 2.1.** *If  $\Sigma_1 \subseteq \Sigma_2 \subseteq Id(\tau)$ ,  $t \in W_\tau(X)$  and  $x_i \in Ess(t, \Sigma_2)$ , then  $x_i \in Ess(t, \Sigma_1)$ .*

**Theorem 2.1.** *Let  $t \in W_\tau(X)$  and  $\Sigma \subseteq Id(\tau)$ . A variable  $x_i$  is  $\Sigma$ -essential for  $t$  if and only if there is a term  $r$  of type  $\tau$  such that  $r \neq x_i$  and  $\mathcal{A} \not\models t \approx t(x_i \leftarrow r)$  for some algebra  $\mathcal{A} \in Alg(\tau)$  with  $\mathcal{A} \models \Sigma$ .*

*Proof.* Let  $t \in W_\tau(X_n)$  for some  $n \in N$  and let  $\mathcal{A} \in Alg(\tau)$  be an algebra for which  $\mathcal{A} \models \Sigma$  and  $x_i \in Ess(t, \mathcal{A})$ . Then from Lemma 3.5 of [5] it follows that  $\mathcal{A} \not\models t \approx t(x_i \leftarrow x_{n+1})$ . Hence  $\mathcal{A} \not\models t \approx t(x_i \leftarrow r)$  with  $r = x_{n+1}$ .

Conversely, let us assume that there is a term  $r$ ,  $r \neq x_i$  of type  $\tau$  with  $\mathcal{A} \not\models t \approx t(x_i \leftarrow r)$  for an algebra  $\mathcal{A} \in Alg(\tau)$  with  $\mathcal{A} \models \Sigma$ .

Let  $m \in N$  be a natural number for which  $r \in W_\tau(X_m)$ . So, there are  $m + n$  values  $a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n, b_1, \dots, b_m \in A$  such that  $r^{\mathcal{A}}(b_1, \dots, b_m) \neq a_i$  and

$$t^{\mathcal{A}}(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \neq t^{\mathcal{A}}(a_1, \dots, a_{i-1}, r^{\mathcal{A}}(b_1, \dots, b_m), a_{i+1}, \dots, a_n).$$

The last inequality shows that  $x_i \in Ess(t, \mathcal{A})$ . Hence  $x_i$  is  $\Sigma$ -essential for  $t$ .  $\square$

**Corollary 2.1.** *If  $t \approx s \in \Sigma$  and  $x_i \in Fic(t, \Sigma)$ , then for each term  $r \in W_\tau(X)$ , we have  $\Sigma \models t(x_i \leftarrow r) \approx s$ .*

**Corollary 2.2.** *A variable  $x_i$  is  $\Sigma$ -essential for  $t \in W_\tau(X_n)$  if and only if  $x_i$  is essential for  $t$  with respect to any  $Mod(\Sigma)$ -free algebra with at least  $n + 1$  free generators.*

**Corollary 2.3.** *Let  $\Sigma \subseteq Id(\tau)$  be a set of identities of type  $\tau$  and  $t \approx s \in \Sigma$ . If a variable  $x_i$  is  $\Sigma$ -fictive for  $t$ , then it is fictive for  $s$  with respect to each algebra  $\mathcal{A} \in Mod(\Sigma)$ .*

The concept of  $\Sigma$ -essential positions is a natural extension of  $\Sigma$ -essential variables.

**Definition 2.4.** Let  $\mathcal{A} = \langle A, \mathcal{F} \rangle$  be an algebra of type  $\tau$ ,  $t \in W_\tau(X_n)$ , and let  $p \in Pos(t)$ .

(i) If  $x_{n+1} \in Ess(t(p; x_{n+1}), \mathcal{A})$ , then the position  $p \in Pos(t)$  is called *essential* for  $t$  with respect to the algebra  $\mathcal{A}$ . The set of all essential positions for  $t$  with respect to  $\mathcal{A}$  is denoted by  $PEss(t, \mathcal{A})$ . When a position  $p \in Pos(t)$  is not essential

for  $t$  with respect to  $\mathcal{A}$ , it is called *fictive* for  $t$  with respect to  $\mathcal{A}$ . The set of all fictive positions with respect to  $\mathcal{A}$  is denoted by  $PFic(t, \mathcal{A})$ .

(ii) If  $x_{n+1} \in Ess(t(p; x_{n+1}), \Sigma)$ , then the position  $p \in Pos(t)$  is called  $\Sigma$ -essential for  $t$ . The set of  $\Sigma$ -essential positions for  $t$  is denoted by  $PEss(t, \Sigma)$ . When a position is not  $\Sigma$ -essential for  $t$  it is called  $\Sigma$ -fictive.  $PFic(t, \Sigma)$  denotes the set of all  $\Sigma$ -fictive positions for  $t$ .

The set of  $\Sigma$ -essential subterms of  $t$  is defined as follows:  $SEss(t, \Sigma) := \{sub_t(p) \mid p \in PEss(t, \Sigma)\}$ .  $SFic(t, \Sigma)$  denotes the set  $SFic(t, \Sigma) := Sub(t) \setminus SEss(t, \Sigma)$ .

So,  $\Sigma$ -essential subterms of a term are subterms which occur at a  $\Sigma$ -essential position. Since one subterm can occur at more than one position in a term, and can occur in both  $\Sigma$ -essential and non- $\Sigma$ -essential positions, we note that a subterm is  $\Sigma$ -essential if it occurs at least once in a  $\Sigma$ -essential position, and  $\Sigma$ -fictive otherwise.

**Example 2.2.** Let  $\tau = (2)$  and let  $t = f(f(x_1, x_2), f(f(x_1, x_2), x_3))$ . Let us consider the variety  $RB = Mod(\Sigma)$  of rectangular bands, where

$$\Sigma = \{f(x_1, f(x_2, x_3)) \approx f(f(x_1, x_2), x_3) \approx f(x_1, x_3), f(x_1, x_1) \approx x_1\}.$$

It is not difficult to see that the  $\Sigma$ -essential positions and subterms of  $t$  are

$$\begin{aligned} PEss(t, \Sigma) &= \{\varepsilon, 1, 11, 2, 22\}, \\ SEss(t, \Sigma) &= \{t, f(x_1, x_2), x_1, f(f(x_1, x_2), x_3), x_3\} \\ PFic(t, \Sigma) &= Pos(t) \setminus PEss(t, \Sigma) = \{12, 21, 211, 212\}, \\ SFic(t, \Sigma) &= \{x_2\}. \end{aligned}$$

The  $\Sigma$ -essential and  $\Sigma$ -fictive positions of  $t$  are represented by large and small black circles, respectively in Figure 2. Note that  $|PFic(t, \Sigma)| > |SFic(t, \Sigma)|$ . This is because there is one subterm,  $f(x_1, x_2)$ , which occurs more than once, once each in an essential and non-essential position, so that  $|Pos(t)| > |Sub(t)|$ .

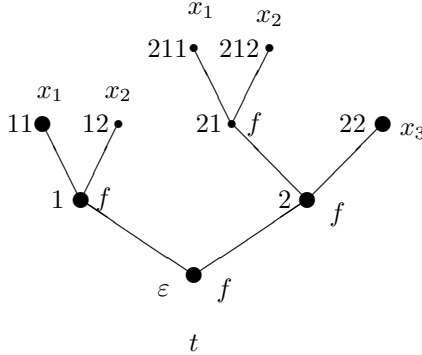


FIGURE 2.  $\Sigma$ -essential positions for  $t$  from Example 2.2.

**Theorem 2.2.** *If  $p \in PEss(t, \Sigma)$ , then each position  $q \in Pos(t)$  with  $q \preceq p$  is  $\Sigma$ -essential for  $t$ .*

*Proof.* Let  $sub_t(q) = s$  and  $sub_t(p) = r$ . Now,  $q \preceq p$  implies that  $r \in Sub(s)$  and  $Sub(r) \subset Sub(s)$ . Let  $n$  be a natural number such that  $t \in W_\tau(X_n)$ .

From  $p \in PEss(t, \Sigma)$  it follows that there is a term  $v \in W_\tau(X)$  for which  $v \neq x_{n+1}$  and

$$\Sigma \not\models t \approx t(p; x_{n+1})(x_{n+1} \leftarrow v), \quad \text{i.e.,} \quad \Sigma \not\models t \approx t(p; v).$$

Consequently, there is an algebra  $\mathcal{A} = \langle A, \mathcal{F} \rangle$  of type  $\tau$  such that

$$\mathcal{A} \models \Sigma \quad \text{and} \quad t^{\mathcal{A}} \neq t(p; v)^{\mathcal{A}}.$$

Let  $m$  be a natural number such that  $t \in W_\tau(X_m)$  and  $v \in W_\tau(X_m)$ .

Let  $(a_1, \dots, a_m) \in A^m$  be a tuple such that

$$t^{\mathcal{A}}(a_1, \dots, a_m) \neq t(p; v)^{\mathcal{A}}(a_1, \dots, a_m).$$

Let  $u \in W_\tau(X_m)$  be the term  $u = s(q'; v)$ , where  $p = q \circ q'$  and  $q' \in Pos(s)$ . Hence we have  $\Sigma \models t(p; v) \approx t(q; u)$  and

$$t^{\mathcal{A}}(a_1, \dots, a_m) \neq t(q; u)^{\mathcal{A}}(a_1, \dots, a_m).$$

Consequently  $t^{\mathcal{A}} \neq t(q; u)^{\mathcal{A}}$ , i.e.,

$$\Sigma \not\models t \approx t(q; x_{n+1})(x_{n+1} \leftarrow u)$$

and  $q \in PEss(t, \Sigma)$ . □

**Corollary 2.4.** *If  $q \in PFic(t, \Sigma)$ , then each position  $p \in Pos(t)$  with  $q \preceq p$  is  $\Sigma$ -fictive for  $t$ .*

**Theorem 2.3.** *Let  $t \in W_\tau(X)$  be a term of type  $\tau$  and let  $\Sigma \subseteq Id(\tau)$  be a set of identities of type  $\tau$ . If  $p \in PFic(t, \Sigma)$ , then  $\Sigma \models t \approx t(p; v)$ , for each term  $v \in W_\tau(X)$ .*

*Proof.* Let  $p \in PFic(t, \Sigma)$  and let us suppose that the theorem is false. Then there is a term  $v \in W_\tau(X)$  with  $v \neq sub_t(p)$ , such that  $\Sigma \not\models t \approx t(p; v)$ . Let  $sub_t(p) = r$  and let  $n$  be a natural number, such that  $v, t \in W_\tau(X_n)$ . Then

$$t(p; v) = t(p; x_{n+1})(x_{n+1} \leftarrow v) \quad \text{and} \quad t = t(p; r) = t(p; x_{n+1})(x_{n+1} \leftarrow r).$$

Our supposition shows that

$$\Sigma \not\models t \approx t(p; v) \iff \Sigma \not\models t(p; x_{n+1})(x_{n+1} \leftarrow r) \approx t(p; x_{n+1})(x_{n+1} \leftarrow v).$$

Hence there is an algebra  $\mathcal{A} = \langle A, \mathcal{F} \rangle$  and  $n + 2$  elements  $a_1, \dots, a_n, a, b$  of  $A$  such that

$$(t(p; x_{n+1}))^{\mathcal{A}}(a_1, \dots, a_n, a) \neq (t(p; x_{n+1}))^{\mathcal{A}}(a_1, \dots, a_n, b),$$

where  $a = r^{\mathcal{A}}(a_1, \dots, a_n)$  and  $b = v^{\mathcal{A}}(a_1, \dots, a_n)$ .

This means that  $x_{n+1} \in Ess(t(p; x_{n+1}), \Sigma)$ . Hence  $p \in PEss(t, \Sigma)$ , which is a contradiction. □

**Corollary 2.5.** *If  $p \in \text{Pos}(t)$  is a  $\Sigma$ -fictive position for  $t$ , then  $p$  is fictive for  $t$  with respect to each algebra  $\mathcal{A}$  with  $\mathcal{A} \models \Sigma$ .*

**Corollary 2.6.** *If  $p \in \text{PEss}(t, \Sigma)$ ,  $t \in W_\tau(X_n)$ , then  $p$  is essential for  $t$  with respect to each  $\text{Mod}(\Sigma)$ -free algebra with at least  $n + 1$  free generators.*

If  $\Sigma \models t \approx s$  and  $s \in \text{Sub}(t)$  is a proper subterm of  $t$ , one might expect that the positions of  $t$  which are “outside” of  $s$  have to be  $\Sigma$ -fictive. To see that this is not true, we consider the set of operations  $\vee, \wedge$  and  $\neg$  with type  $\tau := (2, 2, 1)$ . Let  $\Sigma$  be the set of identities satisfied in a Boolean algebra. Then it is easy to prove that if  $t = x_1 \wedge (x_2 \vee \neg(x_2))$ , then we have  $\Sigma \models t \approx x_1$ , but  $\text{PEss}(t, \Sigma) = \text{Pos}(t)$ .

Now, we are going to generalize composition of terms and to describe the corresponding deductive systems.

Let  $\Sigma$  be a set of identities of type  $\tau$ . Two terms  $t$  and  $s$  are called  $\Sigma$ -equivalent (briefly,  $\Sigma$ -equal) if  $\Sigma \models t \approx s$ .

**Definition 2.5.** Let  $t, r, s \in W_\tau(X)$  and  $\Sigma S_r^t = \{v \in \text{Sub}(t) \mid \Sigma \models r \approx v\}$  be the set of all subterms of  $t$  which are  $\Sigma$ -equal to  $r$ .

Term  $\Sigma$ -composition of  $t$  and  $r$  by  $s$  is defined as follows

- (i)  $t^\Sigma(r \leftarrow s) = t$  if  $\Sigma S_r^t = \emptyset$ ;
- (ii)  $t^\Sigma(r \leftarrow s) = s$  if  $\Sigma \models t \approx r$ , and
- (iii)  $t^\Sigma(r \leftarrow s) = f(t_1^\Sigma(r \leftarrow s), \dots, t_n^\Sigma(r \leftarrow s))$ , if  $t = f(t_1, \dots, t_n)$ .

Let  $\Sigma P_r^t = \{p \in \text{Pos}(t) \mid \text{sub}_t(p) \in \Sigma S_r^t\}$  be the set of all positions of subterms of  $t$  which are  $\Sigma$ -equal to  $r$ . Let  $P_r^t = \{p_1, \dots, p_m\}$  be the set of all the minimal elements in  $\Sigma P_r^t$  with respect to the ordering  $<$  in the set of positions, i.e.,  $p \in P_r^t$  if for each  $q \in P_r^t$  we have  $q \not\prec p$ . Let  $r_j = \text{sub}_t(p_j)$  for  $j = 1, \dots, m$ . Clearly,

$$t^\Sigma(r \leftarrow s) = t(P_r^t, s).$$

**Example 2.3.** Let us consider the set  $\Sigma$  of identities satisfied in the variety  $RB$  of rectangular bands (see Example 2.2). Let  $r = f(x_1, x_2)$  and let the terms  $t$  and  $u$  be the same as in Example 2.1. Then we have

$$\Sigma S_r^t = \{f(x_1, x_2), f(f(x_1, x_2), x_2)\}, \quad \Sigma P_r^t = \{1211, 121\}, \quad P_r^t = \{121\}$$

and  $t^\Sigma(r \leftarrow u) = t(121; u)$  (see Figure 1).

So, the term  $t^\Sigma(r \leftarrow u)$  is the term obtained from  $t$  by replacing  $r$  by  $u$  at any minimal positions whose subterm is  $\Sigma$ -equal to  $r$ , where minimality refers to the order  $\preceq$  on the set of positions.

**Proposition 2.2.** *If  $\Sigma \models r \approx v$ , and  $u, t \in W_\tau(X)$  then:*

- (i)  $\Sigma \models t^\Sigma(u \leftarrow u) \approx t$ ;
- (ii)  $P_r^t = P_v^t$ ;
- (iii)  $\Sigma \models t^\Sigma(r \leftarrow u) \approx t^\Sigma(v \leftarrow u)$ .

*Proof.* (i) If  $t^\Sigma(u \leftarrow u) = t$ , then the proposition is obvious. Let us assume that  $t^\Sigma(u \leftarrow u) \neq t$ . Hence  $\Sigma P_u^t \setminus P_u^t = \{q_1, \dots, q_k\} \neq \emptyset$ . Let  $P_u^t = \{p_1, \dots, p_m\}$  and



$p_i \in P_u^t$ . If  $p_i \not\prec q_j$  for all  $j \in \{1, \dots, k\}$ , then since  $\Sigma \models \text{sub}_t(p_i) \approx u$  and  $D_5$  we obtain  $\Sigma \models t(p_i; u) \approx t$ . If  $p_i \prec q_j$ , for some  $j \in \{1, \dots, k\}$ , then we have  $\Sigma \models \text{sub}_t(p_i) \approx \text{sub}_t(q_j) \approx u$ . Since  $t = t(p_i; \text{sub}_t(p_i)) = t(p_i; \text{sub}_t(p_i))(q_j; \text{sub}_t(q_j))$  we obtain  $\Sigma \models t(p_i; u) \approx t$ . Finally, we have  $\Sigma \models t^\Sigma(u \leftarrow u) \approx t$ .

(ii) and (iii) are clear.  $\square$

**Corollary 2.7.** (i)  $P_u^{t^\Sigma(u \leftarrow u)} = \Sigma P_u^{t^\Sigma(u \leftarrow u)}$ ;  
 (ii)  $t^\Sigma(u \leftarrow u)^\Sigma(u \leftarrow v) = t^\Sigma(u \leftarrow u)(u \leftarrow v)$  for any term  $v \in W_\tau(X)$ .

**Proposition 2.3.** If  $\Sigma \models t \approx s$  and  $\Sigma \models r \approx v$ , then

$$P_r^t \subseteq PFic(t, \Sigma) \iff P_v^s \subseteq PFic(s, \Sigma).$$

Next we consider a deductive system, which is based on the  $\Sigma$ -compositions of terms.

### 3. STABLE VARIETIES AND GLOBALLY INVARIANT CONGRUENCES

Our next goal is to introduce deductive closures on the subsets of  $Id(\tau)$  which generate elements of the lattices  $\mathcal{L}(\tau)$  and  $\mathcal{E}(\tau)$ . These closures are based on two concepts - satisfaction of an identity by a variety and deduction of an identity.

**Definition 3.1.** [1] A set  $\Sigma$  of identities of type  $\tau$  is  $D$ -deductively closed if it satisfies the following axioms (some authors call them “deductive rules”, “derivation rules”, “productions”, etc.):

- $D_1$  (reflexivity)  $t \approx t \in \Sigma$  for each term  $t \in W_\tau(X)$ ;
- $D_2$  (symmetry)  $(t \approx s \in \Sigma) \Rightarrow s \approx t \in \Sigma$ ;
- $D_3$  (transitivity)  $(t \approx s \in \Sigma) \ \& \ (s \approx r \in \Sigma) \Rightarrow t \approx r \in \Sigma$ ;
- $D_4$  (variable inductive substitution)  
 $(t \approx s \in \Sigma) \ \& \ (r \in W_\tau(X)) \Rightarrow t(x \leftarrow r) \approx s(x \leftarrow r) \in \Sigma$ ;
- $D_5$  (term positional replacement)  
 $(t \approx s \in \Sigma) \ \& \ (r \in W_\tau(X)) \ \& \ (\text{sub}_r(p) = t) \Rightarrow r(p; s) \approx r \in \Sigma$ .

For any set of identities  $\Sigma$  the smallest  $D$ -deductively closed set containing  $\Sigma$  is called the  $D$ -closure of  $\Sigma$  and it is denoted by  $D(\Sigma)$ .

Let  $\Sigma$  be a set of identities of type  $\tau$ . For  $t \approx s \in Id(\tau)$  we say  $\Sigma \vdash t \approx s$  (“ $\Sigma$   $D$ -proves  $t \approx s$ ”) if there is a sequence of identities ( $D$ -deduction)  $t_1 \approx s_1, \dots, t_n \approx s_n$ , such that each identity belongs to  $\Sigma$  or is a result of applying any of the derivation rules  $D_1 - D_5$  to previous identities in the sequence and the last identity  $t_n \approx s_n$  is  $t \approx s$ .

According to [1],  $\Sigma \models t \approx s$  if and only if  $t \approx s \in D(\Sigma)$  and the closure  $D(\Sigma)$  is a fully invariant congruence for each set  $\Sigma$  of identities of a given type. It is known that there exists a variety  $V \subset \mathcal{Alg}(\tau)$  with  $Id(V) = \Sigma$  if and only if  $\Sigma$  is a fully invariant congruence (Theorem 14.17 [1]).

Using properties of the essential variables and positions we can divide the rules  $D_4$  and  $D_5$  into four rules which distinguish between operating with essential or fictive objects in the identities.

**Proposition 3.1.** *A set  $\Sigma$  is  $D$ -deductively closed if it satisfies rules  $D_1, D_2, D_3$  and*

$$\begin{aligned}
& D'_4 \text{ (essential variable inductive substitution)} \\
& (t \approx s \in \Sigma) \ \& \ (r \in W_\tau(X)) \ \& \ (x \in \text{Ess}(t, \Sigma)) \Rightarrow t(x \leftarrow r) \approx s(x \leftarrow r) \in \Sigma; \\
& D''_4 \text{ (fictive variable inductive substitution)} \\
& (t \approx s \in \Sigma) \ \& \ (r \in W_\tau(X)) \ \& \ (x \in \text{Fic}(t, \Sigma)) \Rightarrow t(x \leftarrow r) \approx s \in \Sigma; \\
& D'_5 \text{ (essential positional term replacement)} \\
& (t \approx s \in \Sigma) \ \& \ (\text{sub}_r(p) = t, \ p \in \text{PEss}(r, \Sigma)) \Rightarrow r(p; s) \approx r \in \Sigma; \\
& D''_5 \text{ (fictive positional term replacement)} \\
& (t, s, r \in W_\tau(X)) \ \& \ (\text{sub}_r(p) = t, \ p \in \text{PFic}(r, \Sigma)) \Rightarrow r(p; s) \approx r \in \Sigma.
\end{aligned}$$

We will say that a set  $\Sigma$  of identities is complete if  $D(\Sigma) = \text{Id}(\tau)$ . It is clear that if  $\Sigma$  is a complete set, then  $\text{Mod}(\Sigma)$  is a trivial variety.

For fictive positions in terms and complete sets of identities, we have:

$$\Sigma \text{ is complete} \iff (\forall t \in W_\tau(X)) \text{ Pos}(t) = \text{PFic}(t, \Sigma).$$

Let  $\Sigma$  be a non-complete set of identities. Then from Theorem 2.1 and Theorem 2.3, it follows that when applying the rules  $D'_4$  and  $D''_5$ , we can skip these steps in the deduction process, without any reflection on the resulting identities. Hence, if  $\Sigma$  is a non-complete set of identities, then  $\Sigma$  is  $D$ -deductively closed if it satisfies the rules  $D_1, D_2, D_3, D'_4, D'_5$ .

In order to obtain new elements in the lattices  $\mathcal{L}(\tau)$  and  $\mathcal{E}(\tau)$ , we have to extend the derivation rules  $D_1 - D_5$ .

**Definition 3.2.** A set  $\Sigma$  of identities is  $\Sigma R$ -deductively closed if it satisfies the rules  $D_1, D_2, D_3, D_4$  and

$$\begin{aligned}
& \Sigma R_1 \text{ } (\Sigma \text{ replacement}) \\
& (t \approx s, r \approx v, u \approx w \in \Sigma) \ \& \ (r \in \text{SEss}(t, \Sigma)) \ \& \ (v \in \text{SEss}(s, \Sigma)) \\
& \Rightarrow t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w) \in \Sigma.
\end{aligned}$$

For any set of identities  $\Sigma$  the smallest  $\Sigma R$ -deductively closed set containing  $\Sigma$  is called  $\Sigma R$ -closure of  $\Sigma$  and it is denoted by  $\Sigma R(\Sigma)$ .

Let  $\Sigma$  be a set of identities of type  $\tau$ . For  $t \approx s \in \text{Id}(\tau)$  we say  $\Sigma \vdash_{\Sigma R} t \approx s$  (“ $\Sigma$   $\Sigma R$ -proves  $t \approx s$ ”) if there is a sequence of identities  $t_1 \approx s_1, \dots, t_n \approx s_n$ , such that each identity belongs to  $\Sigma$  or is a result of applying any of the derivation rules  $D_1, D_2, D_3, D_4$  or  $\Sigma R_1$  to previous identities in the sequence and the last identity  $t_n \approx s_n$  is  $t \approx s$ .

Let  $t \approx s$  be an identity and  $\mathcal{A}$  be an algebra of type  $\tau$ .  $\mathcal{A} \models_{\Sigma R} t \approx s$  means that  $\mathcal{A} \models t^\Sigma(r \leftarrow v) \approx s^\Sigma(r \leftarrow v)$  for every  $r \in \text{SEss}(t, \Sigma) \cap \text{SEss}(s, \Sigma)$  and  $v \in W_\tau(X)$ .

Let  $\Sigma$  be a set of identities. For  $t, s \in W_\tau(X)$  we say  $\Sigma \models_{\Sigma R} t \approx s$  (read: “ $\Sigma$   $\Sigma R$ -yields  $t \approx s$ ”) if, given any algebra  $\mathcal{A}$ ,

$$\mathcal{A} \models_{\Sigma R} \Sigma \Rightarrow \mathcal{A} \models_{\Sigma R} t \approx s.$$

**Theorem 3.1.**  $\Sigma R$  is a closure operator in the set  $Id(\tau)$ , i.e.,

- (i)  $\Sigma \subseteq \Sigma R(\Sigma)$ ;
- (ii)  $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \Sigma_1 R(\Sigma_1) \subseteq \Sigma_2 R(\Sigma_2)$ ;
- (iii)  $\Sigma R(\Sigma R(\Sigma)) = \Sigma R(\Sigma)$ .

The following lemma is clear.

**Lemma 3.1.** For each set  $\Sigma \subseteq Id(\tau)$  and for each identity  $t \approx s \in Id(\tau)$  we have  $\Sigma \vdash_{\Sigma R} t \approx s \iff \Sigma R(\Sigma) \vdash t \approx s$ .

**Theorem 3.2.** (The Completeness Theorem for  $\Sigma R$ -Equational Logic) Let  $\Sigma \subseteq Id(\tau)$  be a set of identities and  $t \approx s \in Id(\tau)$ . Then

$$\Sigma \models_{\Sigma R} t \approx s \iff \Sigma \vdash_{\Sigma R} t \approx s.$$

*Proof.* The implication  $\Sigma \vdash_{\Sigma R} t \approx s \Rightarrow \Sigma \models_{\Sigma R} t \approx s$  follows by  $\Sigma \vdash_{\Sigma R} t \approx s \Rightarrow t \approx s \in \Sigma R(\Sigma)$  since we have used only properties under which  $\Sigma R(\Sigma)$  is closed, i.e., under  $D_1, D_2, D_3, D_4$  and  $\Sigma R_1$ .

For the converse of this, let us note that for  $t \in W_\tau(X)$  we have  $\Sigma \vdash_{\Sigma R} t \approx t$  and if  $t \approx s \in \Sigma$  then  $\Sigma \vdash_{\Sigma R} t \approx s$ .

If  $\Sigma \vdash_{\Sigma R} t \approx s$ , then there is a formal  $\Sigma R$ -deduction  $t_1 \approx s_1, \dots, t_n \approx s_n$  of  $t \approx s$ . But then  $t_1 \approx s_1, \dots, t_n \approx s_n, s_n \approx t_n$  is a  $\Sigma R$ -deduction of  $s \approx t$ .

If  $\Sigma \vdash_{\Sigma R} t \approx s$  and  $\Sigma \vdash_{\Sigma R} s \approx r$  let  $t_1 \approx s_1, \dots, t_n \approx s_n$  be a  $\Sigma R$ -deduction of  $t \approx s$  and let  $\bar{s}_1 \approx r_1, \dots, \bar{s}_k \approx r_k$  be a  $\Sigma R$ -deduction of  $s \approx r$ . Then

$$t_1 \approx s_1, \dots, t_n \approx s_n, \bar{s}_1 \approx r_1, \dots, \bar{s}_k \approx r_k, t_n \approx r_k$$

is a  $\Sigma R$ -deduction of  $t \approx r$ . Hence  $\Sigma \vdash_{\Sigma R} t \approx r$ .

Let  $\Sigma \models_{\Sigma R} t \approx s$ ,  $\Sigma \models_{\Sigma R} r \approx v$ ,  $\Sigma \models_{\Sigma R} u \approx w$  and  $\Sigma \models_{\Sigma R} t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$ . Suppose that  $\Sigma \vdash_{\Sigma R} t \approx s$ ,  $\Sigma \vdash_{\Sigma R} r \approx v$  and  $\Sigma \vdash_{\Sigma R} u \approx w$ . Let  $t_1 \approx s_1, \dots, t_n \approx s_n$ ,  $r_1 \approx v_1, \dots, r_m \approx v_m$  and  $u_1 \approx w_1, \dots, u_k \approx w_k$  be  $\Sigma R$ -deductions of  $t \approx s$ ,  $r \approx v$  and  $u \approx w$ . Then

$$t_1 \approx s_1, \dots, t_n \approx s_n, r_1 \approx v_1, \dots, r_m \approx v_m, \\ u_1 \approx w_1, \dots, u_k \approx w_k, t_n^\Sigma(r_m \leftarrow u_k) \approx s_n^\Sigma(v_m \leftarrow w_k)$$

is a  $\Sigma R$ -deduction of  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$ . Hence  $\Sigma \vdash_{\Sigma R} t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$ .  $\square$

**Theorem 3.3.** For each set of identities  $\Sigma$  the closure  $\Sigma R(\Sigma)$  is a fully invariant congruence, but  $\Sigma R(\Sigma)$  is not in general equal to  $D(\Sigma)$ .

*Proof.* Let  $\Sigma$  be a  $\Sigma R$ -deductively closed set of identities. We will prove that  $\Sigma$  is a fully invariant congruence. It has to be shown that  $\Sigma$  satisfies the rule  $D_5$ , i.e., if  $r \in W_\tau(X)$ ,  $t \approx s \in \Sigma$  and  $p \in Pos(r)$ , then  $r(p; t) \approx r(p; s) \in \Sigma$ .

If  $p \notin PEss(r, \Sigma)$ , then according to Proposition 3.1 we have  $\Sigma \models r(p; v) \approx r(p; w)$  for all terms  $v, w \in W_\tau(X)$ .

Let  $p \in PEss(r, \Sigma)$  and let  $n$  be a natural number such that  $r, t, s \in W_\tau(X_n)$  and let us consider the term  $u = r(p; x_{n+1})$ . Clearly,  $u \approx u \in \Sigma$ , because of  $D_1$ . We have

$$u^\Sigma(x_{n+1} \leftarrow v) = u(x_{n+1} \leftarrow v) = u(p; v)$$

for each  $v \in W_\tau(X)$ . Now from  $\Sigma R_1$  we obtain

$$u^\Sigma(x_{n+1} \leftarrow t) \approx u^\Sigma(x_{n+1} \leftarrow s) \in \Sigma, \text{ i.e., } u(x_{n+1} \leftarrow t) \approx u(x_{n+1} \leftarrow s) \in \Sigma,$$

and  $r(p; t) \approx r(p; s) \in \Sigma$ .

Furthermore, we will produce a fully invariant congruence  $\Sigma$ , which is not  $\Sigma R$ -deductively closed. Let us consider the variety  $SG = Mod(\Sigma)$  of semigroups, where  $\Sigma = \{f(x_1, f(x_2, x_3)) \approx f(f(x_1, x_2), x_3)\}$ .

From Theorem 14.17 of [1] it follows that if  $\Sigma$  is a fully invariant congruence, then  $D(\Sigma) = Id(Mod(\Sigma))$ . Hence  $Id(SG) = D(\Sigma)$ .

Let

$$t = f(f(f(x_1, x_2), x_1), x_2) \quad \text{and} \quad s = f(f(x_1, x_2), f(x_1, x_2)).$$

It is not difficult to see that  $\Sigma \models t \approx s$ , i.e.,  $t \approx s \in D(\Sigma)$ . Let us set  $r = v = f(x_1, x_2)$  and  $u = w = x_1$ . Clearly, for each  $z \in W_\tau(X)$  we have  $PEss(z, \Sigma) = Pos(z)$  and  $r \in SEss(t, \Sigma)$ , and  $v \in SEss(s, \Sigma)$ . Since  $P_r^t = \{11\}$  and  $P_s^v = \{1, 2\}$ , we obtain

$$t^\Sigma(r \leftarrow u) = f(f(x_1, x_1), x_2) \quad \text{and} \quad s^\Sigma(v \leftarrow w) = f(x_1, x_1).$$

Hence

$$\Sigma \not\models t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w).$$

On the other side we have  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w) \in \Sigma R(\Sigma)$ . Consequently,  $D(\Sigma)$  is a proper subset (equational theory) of  $\Sigma R(\Sigma)$  and  $Mod(\Sigma R(\Sigma))$  is a proper subvariety of  $SG$ , which contains the variety  $RB$  of rectangular bands as a subvariety, according to Example 3.1, below.  $\square$

**Lemma 3.2.** *For each set  $\Sigma \subseteq Id(\tau)$  and for each identity  $t \approx s \in Id(\tau)$  we have  $\Sigma \models_{\Sigma R} t \approx s \iff \Sigma R(\Sigma) \models t \approx s$ .*

The lemma follows immediately from Lemma 3.1 and Theorem 3.2.

**Definition 3.3.** A set of identities  $\Sigma$  is called a globally invariant congruence if it is  $\Sigma R$ -deductively closed.

A variety  $V$  of type  $\tau$  is called stable if  $Id(V)$  is  $\Sigma R$ -deductively closed, i.e.,  $Id(V)$  is a globally invariant congruence.

Note that when  $\Sigma$  is a globally invariant congruence it is possible to apply substitutions or replacements in any place (operation symbol) of terms which explains the word “globally”.

**Example 3.1.** Now, we will produce a fully invariant congruence  $\Sigma$ , which is a globally invariant congruence. Let us consider the variety  $RB = Mod(\Sigma)$  of rectangular bands, where  $\Sigma$  is defined as in Example 2.2.

The set  $\Sigma$  consists of all equations  $s \approx t$  such that the first variable (leftmost) of  $s$  agrees with the first variable of  $t$ , i.e.,  $leftmost(t) = leftmost(s)$  and the last variable (rightmost) of  $s$  agrees with the last variable of  $t$ , i.e.,  $rightmost(t) = rightmost(s)$ . It is well known that  $\Sigma$  is a fully invariant congruence and a totally invariant congruence ( see [2, 4]). From Theorem 14.17 of [1] it follows that  $Id(RB) = D(\Sigma)$ .

Let  $t, s, r, v, u, w \in W_\tau(X)$  be six terms such that  $\Sigma \models t \approx s$ ,  $\Sigma \models r \approx v$ ,  $\Sigma \models u \approx w$ ,  $r \in SEss(t, \Sigma)$  and  $v \in SEss(s, \Sigma)$ .

Thus we have

$$\begin{aligned} leftmost(t) &= leftmost(s), & leftmost(r) &= leftmost(v), \\ leftmost(u) &= leftmost(w), & rightmost(t) &= rightmost(s), \\ rightmost(r) &= rightmost(v), & rightmost(u) &= rightmost(w). \end{aligned}$$

From  $r \in SEss(t, \Sigma)$  and  $v \in SEss(s, \Sigma)$  (see Example 2.2), we obtain

$$\begin{aligned} leftmost(t) &= leftmost(r), & leftmost(s) &= leftmost(v) \text{ or} \\ rightmost(t) &= rightmost(r), & rightmost(s) &= rightmost(v). \end{aligned}$$

Hence

$$\begin{aligned} leftmost(t^\Sigma(r \leftarrow u)) &= leftmost(s^\Sigma(v \leftarrow w)) \text{ and} \\ rightmost(t^\Sigma(r \leftarrow u)) &= rightmost(s^\Sigma(v \leftarrow w)). \end{aligned}$$

We are going to compare globally invariant congruences with the totally invariant congruences, defined by hypersubstitutions.

In [2, 4] the solid varieties are defined by adding a new derivation rule which uses the concept of hypersubstitutions.

Let  $\sigma : \mathcal{F} \rightarrow W_\tau(X)$  be a mapping which assigns to every operation symbol  $f \in \mathcal{F}_n$  an  $n$ -ary term. Such mappings are called *hypersubstitutions* (of type  $\tau$ ). If one replaces every operation symbol  $f$  in a given term  $t \in W_\tau(X)$  by the term  $\sigma(f)$ , then the resulting term  $\hat{\sigma}[t]$  is the image of  $t$  under the extension  $\hat{\sigma}$  on the set  $W_\tau(X)$ . The monoid of all hypersubstitutions is denoted by  $Hyp(\tau)$ .

Let  $\Sigma$  be a set of identities. The hypersubstitution derivation rule is defined as follows:

$$\begin{aligned} H_1 \text{ (hypersubstitution)} \\ (t \approx s \in \Sigma \ \& \ \sigma \in Hyp(\tau)) \Rightarrow \hat{\sigma}[t] \approx \hat{\sigma}[s] \in \Sigma. \end{aligned}$$

A set  $\Sigma$  is called  $\chi$ -deductively closed (*hyperequational theory, or totally invariant congruence*) if it is closed with respect to the rules  $D_1, D_2, D_3, D_4, D_5$  and  $H_1$ . The  $\chi$ -closure  $\chi(\Sigma)$  of a set  $\Sigma$  of identities is defined in a natural way and the meaning of  $\Sigma \models_\chi$  and  $\Sigma \vdash_\chi$  is clear.

It is obvious that  $D(\Sigma) \subseteq \chi(\Sigma)$  for each set of identities  $\Sigma \subseteq Id(\tau)$ . There are examples of  $\Sigma$  such that  $D(\Sigma) \neq \chi(\Sigma)$ , which shows that the corresponding variety  $Mod(\chi(\Sigma))$  is a proper subvariety of  $Mod(D(\Sigma))$ . A variety  $V$  for which  $Id(V)$  is  $\chi$ -deductively closed is called *solid* variety of type  $\tau$  [2].

A more complex closure operator on sets of identities is studied in [3]. This operator is based on the concept of coloured terms and multi-hypersubstitutions.

The next proposition deals with the relations between the closure operators  $\Sigma R$  and  $\chi$ .

**Proposition 3.2.** *There exists a stable variety, which is not a solid variety.*

*Proof.* Let us consider the type  $\tau = (2)$  and  $\Sigma = \{f(f(x_1, x_2), x_1) \approx f(x_1, x_1)\}$ . We will show that  $LA = Mod(\Sigma)$  is a stable variety. So, we have to prove that

$$(1) \quad \Sigma \models t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w),$$

when  $\Sigma \models t \approx s$ ,  $\Sigma \models r \approx v$ ,  $\Sigma \models u \approx w$ ,  $r \in SEss(t, \Sigma)$  and  $v \in SEss(s, \Sigma)$ .

We will proceed by induction on  $Depth(t)$  - the depth of the term  $t$ . The case  $Depth(t) = 0$  is trivial.

Let  $Depth(t) = 1$ . If  $t = f(x_1, x_2)$  then  $\Sigma \models t \approx s$  implies  $s = f(x_1, x_2)$  and (1) is satisfied in this case. Let us consider the case  $t = f(x_1, x_1)$ . If  $s = f(x_1, x_1)$  then clearly (1) holds.

Let  $s = f(f(x_1, s_1), x_1)$  for some  $s_1 \in W_\tau(X)$ . Since the positions in  $s_1$  are  $\Sigma$ -fictive in  $s$  it follows that  $v$  can be one of the terms  $x_1$  or  $s$ . On the other side we have  $SEss(t, \Sigma) = \{x_1, t\}$ . Hence (1) is satisfied, again.

Our inductive supposition is that if  $Depth(t) < k$  then (1) is satisfied for all  $s, r, u, v, w \in W_\tau(X)$  with  $\Sigma \models t \approx s$ ,  $\Sigma \models r \approx v$ ,  $\Sigma \models u \approx w$ ,  $r \in SEss(t, \Sigma)$  and  $v \in SEss(s, \Sigma)$ .

Let  $Depth(t) = k > 1$  and  $Depth(s) \geq k$ . Then we have  $t = f(t_1, t_2)$  and  $s = f(s_1, s_2)$ , such that  $t_1$  or  $t_2$  is not a variable.

If  $\Sigma \models t \approx r$  or  $\Sigma \models s \approx v$ , then by Definition 2.5 (ii) and the transitivity  $D_3$ , it follows that (1) is  $\Sigma \models u \approx w$  and we are done.

Next, we assume that  $\Sigma \not\models t \approx r$  and  $\Sigma \not\models s \approx v$ .

First, let  $t_1 \in X$ . Then  $s_1 = t_1$  and  $\Sigma \models t_2 \approx s_2$ . Thus, from the inductive supposition it follows that (1) is satisfied.

Second, let  $t_1 \notin X$ . Then we have  $s_1 \notin X$ , also. Hence  $t = f(f(t_{11}, t_{12}), t_2)$ ,  $s = f(f(s_{11}, s_{12}), s_2)$  and  $\Sigma \models t_2 \approx s_2$ .

Let  $\Sigma \models t_{11} \approx t_2$  and  $\Sigma \models s_{11} \approx s_2$ . Then we have  $\Sigma \models t \approx f(t_2, t_2)$  and  $\Sigma \models f(t_2, t_2) \approx f(s_2, s_2)$ . On the other side all positions in  $t_{12}$  and  $s_{12}$  are  $\Sigma$ -fictive in  $t$  and  $s$ , respectively. Thus we have

$$\Sigma \models t^\Sigma(r \leftarrow u) \approx f(t_2^\Sigma(r \leftarrow u), t_2^\Sigma(r \leftarrow u)) \quad \text{and}$$

$$\Sigma \models s^\Sigma(v \leftarrow w) \approx f(s_2^\Sigma(v \leftarrow w), s_2^\Sigma(v \leftarrow w)).$$

Hence (1) is satisfied, in this case, again. If  $\Sigma \models t_{11} \approx t_2$  and  $\Sigma \not\models s_{11} \approx s_2$  then we have  $\Sigma \models f(t_2, t_2) \approx f(s_1, s_2)$  and  $\Sigma \models s_1 \approx s_2 \approx t_2$ . This implies that (1) is satisfied, again.

Let  $\Sigma \not\models t_{11} \approx t_2$  and  $\Sigma \models t_1 \approx t_2$ . Then we have  $\Sigma \models t \approx f(t_1, t_2)$ . Now, we proceed similarly as in the case  $\Sigma \models t_{11} \approx t_2$  and  $\Sigma \not\models s_{11} \approx s_2$ . If  $\Sigma \not\models t_{11} \approx t_2$

and  $\Sigma \not\models t_1 \approx t_2$ , then we have  $\Sigma \not\models s_1 \approx s_2$ . Hence  $\Sigma \models t_1 \approx s_1$  and  $\Sigma \models t_2 \approx s_2$ . Again, from the inductive supposition we prove (1).

To prove that  $LA$  is not a solid variety, let us consider the following terms  $y = f(f(x_1, x_2), x_1)$  and  $z = f(x_1, x_1)$ . Let  $\sigma \in Hyp(\tau)$  be the hypersubstitution, defined as follows:  $\sigma(f(x_1, x_2)) := f(x_2, x_1)$ . It is clear that  $\Sigma \models y \approx z$ . On the other side we have  $\hat{\sigma}[y] = f(x_1, f(x_2, x_1))$  and  $\hat{\sigma}[z] = f(x_1, x_1)$ . Thus, we obtain  $\Sigma \not\models \hat{\sigma}[y] \approx \hat{\sigma}[z]$ . Hence  $LA$  is not a solid variety.  $\square$

**Remark 3.1.** By analogy, it follows that the variety

$$RA = Mod(\{f(x_1, f(x_2, x_1)) \approx f(x_1, x_1)\})$$

is stable, but not solid, also.

The varieties of left-zero bands  $L0 = Mod(\{f(x_1, x_2) \approx x_1\})$  and of right-zero bands  $R0 = Mod(\{f(x_1, x_2) \approx x_2\})$  are other examples of stable varieties, which are not solid ones.

We do not know whether there is a non-trivial solid variety which is not stable?

#### 4. $\Sigma$ -BALANCED IDENTITIES AND SIMPLIFICATION OF DEDUCTIONS

Regular identities [1, 4] are identities in which the same variables occur on each side of the identity. Balanced identities are identities in which each variable occurs the same number of times on each side of the identity.

In an analogous way we consider the concept of  $\Sigma$ -balanced identities.

Let  $t, r \in W_\tau(X)$  be two terms of type  $\tau$  and  $\Sigma \subset Id(\tau)$  be a set of identities.  $EP_r^t$  denotes the set of all  $\Sigma$ -essential positions from  $P_r^t$ , i.e.,  $EP_r^t = PEss(t, \Sigma) \cap P_r^t$ .

**Definition 4.1.** Let  $\Sigma \subset Id(\tau)$ . We will say that an identity  $t \approx s$  of type  $\tau$  is  $\Sigma$ -balanced if  $|EP_q^t| = |EP_q^s|$  for all  $q \in W_\tau(X)$ .

**Example 4.1.** Let  $\Sigma$  be the set of identities satisfied in the variety  $RB$  of rectangular bands (see Example 2.2).

Let us consider the following three terms  $t = f(f(x_1, x_2), f(x_1, x_3))$ ,  $s = f(x_1, f(f(x_1, x_2), x_3))$  and  $r = f(f(f(x_1, f(x_3, x_2)), x_3), f(x_1, x_3))$ . Clearly,  $\Sigma \models t \approx s$ ,  $\Sigma \models t \approx r$ ,  $SEss(t, \Sigma) = SEss(r, \Sigma) = \{x_1, f(x_1, x_2), f(x_1, x_3), x_3\}$  and  $SEss(s, \Sigma) = \{x_1, f(x_1, x_3), x_3\}$ . Thus we have  $EP_{x_1}^t = \{11\}$ ,  $EP_{x_3}^t = \{22\}$ ,  $EP_{f(x_1, x_2)}^t = \{1\}$ ,  $EP_{f(x_1, x_3)}^t = \{\varepsilon\}$ ,  $EP_t^t = \{\varepsilon\}$ ,  $EP_{x_1}^s = \{1\}$ ,  $EP_{x_3}^s = \{22\}$ ,  $EP_{f(x_1, x_3)}^s = \{\varepsilon\}$ ,  $EP_s^s = \{\varepsilon\}$ , and  $EP_{x_1}^r = \{111\}$ ,  $EP_{x_3}^r = \{22\}$ ,  $EP_{f(x_1, x_2)}^r = \{11\}$ ,  $EP_{f(x_1, x_3)}^r = \{\varepsilon\}$ ,  $EP_r^r = \{\varepsilon\}$ . Hence the identity  $t \approx r$  is  $\Sigma$ -balanced, but  $t \approx s$  is not  $\Sigma$ -balanced.

**Theorem 4.1.** Let  $\Sigma \subset Id(\tau)$  be a set of  $\Sigma$ -balanced identities. If there is a  $\Sigma R$ -deduction of  $t \approx s$  with  $\Sigma$ -balanced identities, then  $t \approx s$  is a  $\Sigma$ -balanced identity of type  $\tau$ .

*Proof.* Let  $t, s, r \in W_\tau(X)$  and let  $t \approx s$  and  $s \approx r$  be two  $\Sigma$ -balanced identities of type  $\tau$ . Then for each term  $q \in W_\tau(X)$  we have

$$|EP_q^t| = |EP_q^s| \quad \text{and} \quad |EP_q^s| = |EP_q^r|.$$

Hence  $|EP_q^t| = |EP_q^r|$  which shows that the identity  $t \approx r$  is  $\Sigma$ -balanced, too.

Let  $t \approx s$  is a  $\Sigma$ -balanced identity in  $\Sigma R(\Sigma)$  and let  $r \in W_\tau(X)$  be a term with  $t \in SEss(r, \Sigma)$  and  $sub_r(p) = t$ . We have  $\Sigma \models_{\Sigma R} r(p; s) \approx r$ . From Proposition 2.2, we obtain

$$(EP_q^t = EP_q^s \ \& \ t \in SEss(r, \Sigma)) \Rightarrow EP_q^r = EP_q^{r(p; s)}$$

for all  $q \in W_\tau(X)$ . Consequently the identity  $r(p; s) \approx r$  is  $\Sigma$ -balanced, too.

Let  $t \approx s$ ,  $r \approx v$  and  $u \approx w$  be  $\Sigma$ -balanced identities from  $\Sigma R(\Sigma)$ . We have to prove that the resulting identity  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$  is  $\Sigma$ -balanced.

This will be done by induction on the depth (also called “height” by some authors).

(i) The basis of induction is  $Depth(t) = 1$  (the case  $Depth(t) = 0$  is trivial). Let  $t = f(x_1, \dots, x_n) \in W_\tau(X_n)$  and let  $s = g(s_1, \dots, s_m)$ . Hence, if  $r = x_i$  for some  $i \in \{1, \dots, n\}$ , then  $EP_r^t = \{i\}$  and  $|EP_r^t| = |EP_v^s| = 1$ .

(ia) If  $\Sigma \models r \approx t$ , then  $\Sigma \models v \approx s$  and we have  $t^\Sigma(r \leftarrow u) = u$  and  $s^\Sigma(v \leftarrow w) = w$ . Hence  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$  is  $\Sigma$ -balanced in this case.

(ib) Let  $\Sigma \not\models r \approx x_i$  for each  $x_i \in X_n$ , i.e.,  $r \notin X_n$  and  $\Sigma \not\models r \approx t$ . If  $r \notin X_n \cup \{t\}$ , then  $EP_r^t = EP_v^s = \emptyset$ . Thus we have

$$t^\Sigma(r \leftarrow u) = t \quad \text{and} \quad s^\Sigma(v \leftarrow w) = s$$

and the resulting identity  $t \approx s$  is  $\Sigma$ -balanced.

(ic) Let  $\Sigma \models r \approx x_i$  for some  $x_i \in X_n$  and  $\Sigma \not\models r \approx t$ . We have

$$t^\Sigma(r \leftarrow u) = t^\Sigma(x_i \leftarrow u) \quad \text{and} \quad s^\Sigma(v \leftarrow w) = s^\Sigma(x_i \leftarrow w).$$

Then  $EP_q^{t^\Sigma(x_i \leftarrow u)} = EP_q^u$  and  $EP_q^{s^\Sigma(x_i \leftarrow w)} = EP_q^w$  for each  $q \in W_\tau(X)$ , i.e., the resulting identity  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$  is  $\Sigma$ -balanced, again.

(ii) Let  $Depth(t) > 1$ ,  $t = f(t_1, \dots, t_n)$  and  $s \in W_\tau(X)$ , such that the identity  $\Sigma \models t \approx s$  is  $\Sigma$ -balanced. Suppose that for each  $t' \in SEss(t, \Sigma)$  with  $t' \neq t$  the following is true: if  $\Sigma \models t' \approx s'$  is  $\Sigma$ -balanced identity for some  $s' \in W_\tau(X)$ , then  $t'^\Sigma(r \leftarrow u) \approx s'^\Sigma(v \leftarrow w)$  is  $\Sigma$ -balanced, also.

(iia) Let  $\Sigma \models r \approx t$ . Then we have

$$EP_q^{t^\Sigma(x_i \leftarrow u)} = EP_q^u \quad \text{and} \quad EP_q^{s^\Sigma(x_i \leftarrow w)} = EP_q^w$$

for each  $q \in W_\tau(X)$  and the resulting identity is  $\Sigma$ -balanced in that case, again.

(iib) Let  $\Sigma \not\models r \approx t$  and  $\Sigma \models r \approx t_i$  for some  $i = 1, \dots, n$ . As in the case (ic) it can be proved that the identity  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$  is  $\Sigma$ -balanced.

(iic) Let  $\Sigma \not\models r \approx t$ ,  $\Sigma \not\models r \approx t_i$  for each  $i = 1, \dots, n$  and there is  $j \in \{1, \dots, n\}$  with  $EP_r^{t_j} = \{r_{j1}, \dots, r_{jk_j}\} \neq \emptyset$ . Without loss of generality assume that all such  $j$  are the natural numbers from the set  $L = \{1, \dots, l\}$  with  $l \leq n$ .



Let  $j \in L$ . If  $\Sigma \models t \approx t_j$  is a  $\Sigma$ -balanced identity, then  $\Sigma \models s \approx t_j$  is  $\Sigma$ -balanced, also and by our assumption, we have that  $t_j^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$  is  $\Sigma$ -balanced identity. Hence we have  $|EP_q^t| = |EP_q^{t_j}|$  for all  $q \in W_\tau(X)$ . This implies that the resulting identity is  $\Sigma$ -balanced in this case, also.

If  $\Sigma \not\models t \approx t_j$  for all  $j \in L$  then since  $t \approx s$  is a  $\Sigma$ -balanced identity, there are subterms  $s_1, \dots, s_l$  of  $s$  such that  $\Sigma \models t_j \approx s_j$ . According to our inductive supposition the last identities are  $\Sigma$ -balanced. Consequently,

$$|EP_r^{t_j}| = |EP_r^{s_j}|, \quad EP_r^t = \cup_{j=1}^l EP_r^{t_j} \quad \text{and} \quad EP_v^s = \cup_{j=1}^l EP_v^{s_j}.$$

Hence  $t^\Sigma(r \leftarrow u) \approx s^\Sigma(v \leftarrow w)$  is a  $\Sigma$ -balanced identity.  $\square$

The complexity of the problem of deduction depends on the complexity of the algorithm for checking when a position of a term is essential or not with respect to a set of identities. The complexity of that algorithm for finite algebras is discussed in [5], but it is based on the full exhaustion of all possible cases.

There should be a case or cases, when the process of deduction can be effectively simplified. This is, for instance, when a variable  $x$  does not belong to  $\text{var}(t)$  and  $\Sigma \models t \approx s$ . Therefore we obtain  $x \notin \text{Ess}(t, \Sigma) \cup \text{Ess}(s, \Sigma)$  (see Theorem 2.1). Then we can skip the rules  $D_4''$  and  $D_5''$ , according to Proposition 3.1. Obviously, it is very easy to check if  $x \in \text{var}(t)$  or not.

#### REFERENCES

- [1] S. Burris, H. Sankappanavar, *A Course in Universal Algebra*, The millennium edition, 2000
- [2] K.Denecke, D.Lau, R.Pöschel, D.Schweigert, *Solidifiable Clones*, General Algebra 20, Heldermann Verlag, Berlin 1993, pp.41-69.
- [3] K. Denecke, J. Koppitz, Sl. Shtrakov, *Multi-Hypersubstitutions and Coloured Solid Varieties*, J. Algebra and Computation, Volume 16, Number 4, August, 2006, pp.797-815.
- [4] E. Graczyńska, *On Normal and Regular Identities and Hyperidentities*, Universal and Applied Algebra, Turawa, Poland 3 - 7 May 1988, World Scientific (1989), 107-135.
- [5] Sl. Shtrakov, K. Denecke, *Essential Variables and Separable Sets in Universal Algebra*, J. Multi. Val. Logic, 2002, vol. 8(2), pp 165-181.
- [6] W. Taylor, *Hyperidentities and Hypervarieties*, Aequationes Mathematicae, 23(1981), 30-49.  
E-mail address: shtrakov@aix.swu.bg

DEPARTMENT OF COMPUTER SCIENCE, SOUTH-WEST UNIVERSITY, 2700 BLAGOEVGRAD, BULGARIA

URL: <http://home.swu.bg/shtrakov>